Ch. 3 Vector-valued functions of one variable

Sec 3.2 Tangents, Velocity and Acceleration

- Def (Differentiation and integration)

For a vector-valued function $\vec{c}(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$,

$c'(t) = (x'(t), y'(t), z'(t))$ and

$\int \vec{c}(t)\, dt = \left(\int x(t)\, dt, \int y(t)\, dt, \int z(t)\, dt \right)$.

Example Find a parametrization of the tangent line to $\vec{c}(t) = (2t, t^3, 0)$ at the point $(4,8,0)$.

Soln Notice that $\vec{c}(2) = (4,8,0)$ so the tangent line has slope $\vec{c}'(2) = (2, 3t^2, 0)\big|_{t=2} = (2,12,0)$.

$\rightarrow \ell(t) = (4,8,0) + t(2,12,0)$.

Example If $\vec{a}(t) = (t, t^2, t)$
$\vec{v}(0) = (0,2,-3)$
$\vec{c}(0) = (4,2,-6)$

then $\vec{v}(t) = \int \vec{a}(t)\, dt = \left(\frac{t^2}{2} + A, \frac{t^3}{3} + B, \frac{t^2}{2} + C \right)$

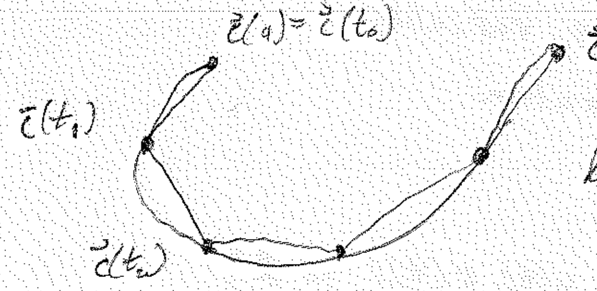
$= \left(\frac{t^2}{2}, \frac{t^3}{3} + 2, \frac{t^2}{2} - 3 \right)$, using $\vec{v}(0)$,

so $\vec{c}(t) = \int \vec{v}(t)\, dt = \left(\frac{t^3}{6} + D, \frac{t^4}{12} + 2t + E, \frac{t^3}{6} - 3t + F \right)$

$= \left(\frac{t^3}{6} + 4, \frac{t^4}{12} + 2t + 2, \frac{t^3}{6} - 3t - 6 \right)$
using $\vec{c}(0)$.

Sec 3.3 Length of a curve

- Def A function is C^1 or continuously differentiable if all of its partial derivatives exist and are continuous.



$$\text{length} \approx \sum_{i=1}^{n} \|\vec c(t_i)-\vec c(t_{i-1})\|$$

$$= \sum_{i=1}^{n} \left\| \frac{\vec c(t_i)-\vec c(t_{i-1})}{t_i - t_{i-1}} \right\| \|t_i - t_{i-1}\|$$

$$\approx \sum_{i=1}^{n} \|c'(t_i)\| \Delta t_i$$

<u>Def</u> Let $\vec c : [a,b] \longrightarrow \mathbb{R}^2$ (or $\mathbb{R}^3$) be C^1. Then

$$\boxed{\text{length}(\vec c) = \ell(\vec c) = \int_a^b \|c'(t)\|\, dt}$$

<u>Example</u> $\vec c_1(t) = (r\cos t, r\sin t)$, $t \in [0, 2\pi]$

$$\ell(\vec c_1) = \int_0^{2\pi} \sqrt{(r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= \int_0^{2\pi} r\, dt = 2\pi r.$$

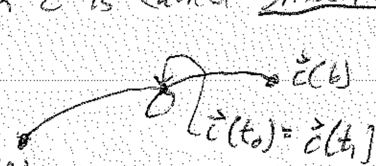
For $\vec c_2(t) = (r\cos t, r\sin t)$, $t \in [0, 4\pi]$, we have $\ell(\vec c_2) = 4\pi r$

Both paths parametrize the same curve but only $\ell(\vec c_1)$ gave the circumference.

- So for $\ell(\vec c)$, think distance traveled (in general).

- <u>Def</u> If $\vec c$ is C^1, $c'(t) \neq \vec 0$, and $\vec c(t)$ maps distinct points in (a,b) to distinct points on the curve, then $\vec c$ is called <u>smooth</u>.

- <u>Def</u> A path is <u>closed</u> if $\vec c(a) = \vec c(b)$.



- A smooth path can be closed, but can't intersect, retrace itself, or have corners.

$\vec c(t) = (\cos^3 t, \sin^3 t)$ has

$c'(0) = \vec 0$

<u>Example</u> Parametrize the graph of a C^1 function $f:[a,b]\to\mathbb{R}$ by $\vec{c}(t) = (t, f(t))$, $t\in[a,b]$. Then $\vec{c}$ is smooth and

$$l(\vec{c}) = \int_a^b \|\vec{c}\,'(t)\|\,dt = \int_a^b \sqrt{1+f'(t)^2}\,dt = \text{3B formula.}$$

• <u>Remark</u> If $\vec{c}$ is smooth, then $l(\vec{c})$ is what one might expect, and not just distance traveled.

• <u>Def</u> Let $\vec{c}:[a,b]\to\mathbb{R}^2$ (or $\mathbb{R}^3$) be C^1. The <u>arc length function</u> $s(t)$ for $\vec{c}(t)$ is:
$$s(t) = \int_a^t \|\vec{c}\,'(\tau)\|\,d\tau = \text{distance traveled in time } t.$$

• <u>Note</u> By the FTC $\dfrac{ds}{dt} = \dfrac{d}{dt}\left(\int_a^t \|\vec{c}\,'(\tau)\|\,d\tau\right) = \|\vec{c}\,'(t)\|$

i.e., $\dfrac{\text{change in distance}}{\text{change in time}} = \text{speed.}$

• If we solve for t in terms of s, then $\vec{c} = \vec{c}(t(s))$ is said to be <u>parametrized by arc length</u>.

<u>Example</u> $\vec{c}(t) = (\cos t, \sin t, t)$, $t\in[0,2\pi]$, has arc length function

$$s(t) = \int_0^t \sqrt{(-\sin\tau)^2+(\cos\tau)^2+1^2}\,d\tau = \int_0^t \sqrt{2}\,d\tau = \sqrt{2}\,t$$

so that $t = t(s) = \dfrac{s}{\sqrt{2}}$, and

$$\vec{c}(t) = \vec{c}(t(s)) = \vec{c}\left(\frac{s}{\sqrt{2}}\right) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right),\ s\in[0, 2\sqrt{2}\pi]$$

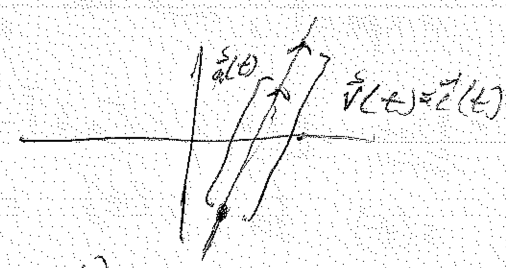
<u>Use</u> After traveling along the curve a distance of $s = \sqrt{2}\pi$, we're at the point $\left(\cos\frac{\sqrt{2}\pi}{\sqrt{2}}, \sin\frac{\sqrt{2}\pi}{\sqrt{2}}, \frac{\sqrt{2}\pi}{\sqrt{2}}\right)$

$$= (-1, 0, \pi).$$

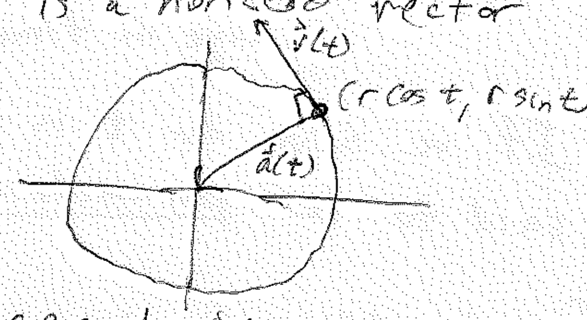
Sec 3.4 Acceleration and curvature

Example For the line $\vec{c}(t) = (1,-3) + t^3(1,2) = (1+t^3, -3+2t^3)$,
$\vec{c}'(t) = (3t^2, 6t^2) = 3t^2(1,2)$ is not constant
(but its direction is)

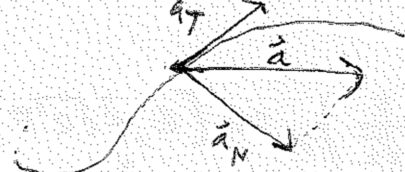
$\vec{c}''(t) = \vec{a}(t) = (6t, 12t) = 6t(1,2)$ is a nonzero vector parallel to the motion given by $\vec{c}'(t)$.



Example For the circle $\vec{c}(t) = (r\cos t, r\sin t)$
$c'(t) = v(t) = (-r\sin t, r\cos t)$, so
the speed $\|c'(t)\| = r$ is constant.
$\vec{c}''(t) = \vec{a}(t) = (-r\cos t, -r\sin t)$ is a nonzero vector perpendicular to the motion.



These are the extreme cases. In general, $\vec{a}(t)$ has a component parallel to the motion, $\vec{a}_T$, and a component perpendicular to the motion, $\vec{a}_N$, so that $\vec{a} = \vec{a}_T + \vec{a}_N$



The <u>tangential component</u> $\vec{a}_T$ corresponds to changes in speed.
The normal component $\vec{a}_N$ corresponds to changes in direction. (as in the examples)

<u>Curvature</u> : a_N is related to motion around a circle.
Goal: Find a circle that best approximates the motion near some point.

$K = \frac{1}{r}$.

- Def The unit tangent vector is $\vec{T}(t) = \frac{c'(t)}{\|c'(t)\|}$.
- Def The curvature of a smooth curve $\vec{c}$ is

$$K = \left\| \frac{dT}{ds} \right\|,$$ where T is the unit tangent vector, and s is the arc length.

In words, the magnitude of the rate of change of the unit tangent vector with respect to arc length.

- A more practical formula is

$$K(t) = \frac{\|\vec{c}'(t) \times \vec{c}''(t)\|}{\|\vec{c}'(t)\|^3} = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|}.$$

Example $\vec{c}(t) = (r\cos t, r\sin t, 0)$.

$$\vec{c}'(t) = (-r\sin t, r\cos t, 0).$$
$$\vec{c}''(t) = (-r\cos t, -r\sin t, 0).$$

$$c' \times c'' = \begin{vmatrix} i & j & k \\ -r\sin t & r\cos t & 0 \\ -r\cos t & -r\sin t & 0 \end{vmatrix} = (0, 0, r^2).$$

so $\|c' \times c''\| = r^2$.

$\|c'(t)\| = r$, so $K(t) = \frac{\|c' \times c''\|}{\|c'\|^3} = \frac{r^2}{r^3} = \frac{1}{r}$

 $K = \frac{1}{r}$ big  $K = \frac{1}{R}$ small

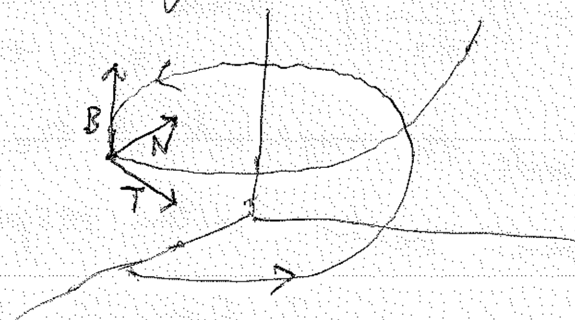
Sec 3.5 Intro to Differential Geometry

- Def The unit normal vector is $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$
- Claim: $\vec{N}(t)$ is orthogonal to $\vec{T}(t)$, and hence the curve.

proof: $\vec{T}(t) \cdot \vec{T}(t) = \|\vec{T}(t)\|^2 = 1 \longrightarrow 2\vec{T}(t) \cdot \vec{T}'(t) = 0$ by the product rule.
so $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$, and so $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$ is as well.

<u>Def</u> The <u>binormal vector</u> $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ is orthogonal to both $\vec{T}$ and $\vec{N}$ and has unit length.

- $\vec{T}$, $\vec{N}$, and $\vec{B}$ form a coordinate system called the <u>Frenet frame</u> at $\vec{c}(t)$.



- Differential geometry application: Analyzing derivatives of $\vec{T}(t)$, $\vec{N}(t)$, $\vec{B}(t)$, which depend on curvature. Motion along a curve studied in terms of how $\vec{T}, \vec{N}, \vec{B}$ change. Application: roller coasters.